DOCUMENT RESUME

ED 218 080 SE 037 818

AUTHOR Cohen, Simon
TITLE Ascent-Descent. Applications of Calculus to Physics and Engineering. Modules and Monographs in Undergraduate Mathematics and Its Applications. UMAP Module 331.
INSTITUTION Education Development Center, Inc., Newton, Mass.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 79
GRANT SED-76-19615-A02
NOTE 17p.

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS *College Mathematics; *Engineering; Graphs; Higher Education; Instructional Materials; *Learning Modules; *Mathematical Applications; *Mathematical Models; *Physics; Problem Solving; Science Instruction; Supplementary Reading Materials

ABSTRACT

Models of the motion of an object through space are discussed. The material begins with the simplest view possible in which gravity is the only force examined. Discussion progresses to an examination of the drag force, which is the resistive air force of the wind. Suggestions for further improvements are made at the conclusion of the presentation. The module contains exercises, and solutions to these problems are provided. (MP)

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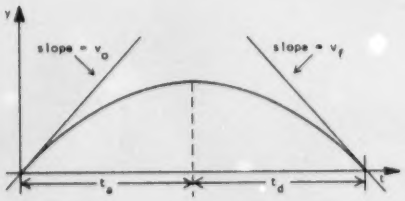
Ascent-Descent

Simon Cohen



Applications of Calculus to Physics and Engineering

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ASCENT - DESCENT

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TABLE OF CONTENTS

1. INTRODUCTION 90

1.1 The Simplest Model 90

1.2 Our Objective 91

2. THE NEW MODEL 91

2.1 The Drag Force 91

2.2 The Equation of Motion 92

2.3 Ascent Time Versus Descent Time 93

2.4 Initial Velocity Versus Final Velocity . . . 94

3. CONCLUDING REMARKS 96

3.1 A Comparison of the Two Models 96

3.2 The Next Step 97

4. ANSWERS TO EXERCISES 98

SPECIAL ASSISTANCE SUPPLEMENT 100

Intermodular Description Sheet: UMAP Unit 331

Title: ASCENT - DESCENT

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Review Stage/Date: III 1/26/79

Classification: APPL CALC/PHYSICS & ENG

Prerequisite Skills:
1. Know the meaning of the terms "model," "differential equation," and "solution to a differential equation."
2. Know Newton's second law of motion.
3. Know the uses of the derivative in curve sketching.
4. Familiarity with exponential and logarithmic functions.

Output Skills:
1. To acquaint the student with the ideas of model construction, model analysis, and model comparison.
2. To present the student with problems whose solutions are obtained by indirect means.

The Project would like to thank Charles Heiberg of the U.S. Naval Academy, Harvey S. Picker of Trinity College, and George C.T. Kung of the University of Wisconsin at Stevens Point for their reviews, and all others who assisted in the production of this unit.

This material was prepared with the partial support of National Science Foundation Grant No. SED76-19615 A02. Recommendations expressed are those of the author and do not necessarily reflect the views of the NSF or of the copyright holder.

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1. INTRODUCTION

1.1 The Simplest Model

Perhaps you are acquainted with the model for the motion of a ball projected vertically upward in which gravity is the only force assumed to be present. In this formulation Newton's second law of motion yields the differential equation

$$(1) \qquad m\ddot{y} = -mg,$$

where m is the mass of the ball, y is the displacement of the ball from ground level, and g is the gravitational acceleration constant (approximately 9.8 m/sec^2). Upon antidifferentiating in (1) twice and using the initial conditions $\dot{y}(0) = v_o$, $y(0) = 0$, we obtain

$$(2) \qquad y = -\frac{1}{2}gt^2 + v_o t.$$

From the symmetry of the parabolic graph of y (see Figure 1) it can be concluded that

A. The ascent time t_a is equal to the descent time t_d.

B. The initial and final velocities, v_o and v_f, are equal in magnitude.

Exercises

1. Starting with Equation (1), carry out in detail the steps that lead to Equation (2).

2. Find the common value of t_a and t_d.

3. Find the maximum height the ball attains.

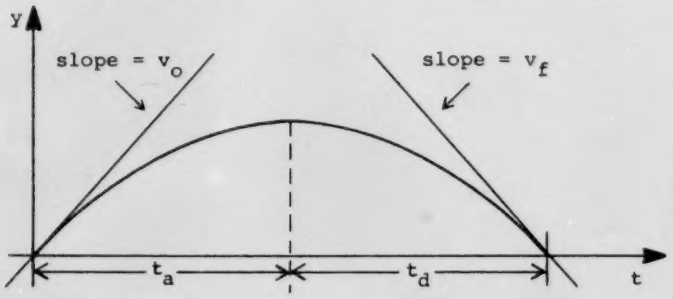


Figure 1. Displacement of graph in the gravity-only case.

1.2 Our Objective

The model presented above is rather primitive and highly idealized. While it did enable us to obtain the relevant parameters of the motion (e.g., A and B, Exercises 2 and 3) without too much difficulty, its basic assumption that gravity is the only force acting on the ball leaves a lot to be desired. These results should therefore be regarded only as first approximations to reality. To obtain a more accurate picture of the motion of the ball we have to use a more sophisticated model. One such model takes the resistive force of the air (the so-called drag force) into account. In this unit we shall investigate this new model and draw conclusions analogous to A and B.

2. THE NEW MODEL

2.1 The Drag Force

Just how do we incorporate the drag force into our model? One possible contributing factor which comes to mind as affecting this force is the velocity of the ball. Experiments have shown that for velocities up to about 100 m/sec the drag force is (approximately) proportional to the velocity. (See Figure 2.) That is, we shall assume

(3) $$D = -k\dot{y}$$

in our model, where k is a positive constant (sometimes referred to as the drag coefficient).

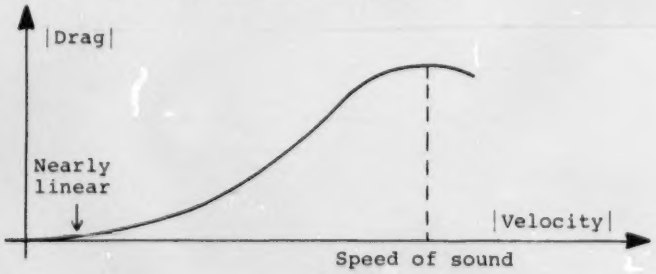


Figure 2. The drag force.

Exercises

4. Explain the use of a minus sign in Equation (3).

5. What other factors, besides the velocity of the ball, might influence the drag force?

2.2 The Equation of Motion

Treating the drag force as indicated in Equation (3), and making use of Newton's second law of motion, gives us

$$m\ddot{y} = -k\dot{y} - mg \tag{4}$$

as our new equation of motion. Dividing through by m and rewriting this equation in terms of the velocity v (= $\dot{y}$), we obtain

$$\dot{v} = \frac{-kv}{m} - g. \tag{5}$$

It is shown in Section [S-1] of the Special Assistance Supplement that all solutions to a differential equation of the type

$$\frac{dx}{dt} = k_1 x + k_2, \tag{6}$$

where k_1 and k_2 are constants ($k_1 \neq 0$), are of the form

$$x = Ae^{k_1 t} - k_2/k_1, \tag{7}$$

where A is an arbitrary constant. Applying this result to Equation (5) (with v playing the role of x, $k_1 = -k/m$, $k_2 = -g$) and using the initial condition $v(0) = v_o$ enables us to conclude that

$$v = \frac{1}{k}(kv_o + mg)e^{-kt/m} - \frac{mg}{k}. \tag{8}$$

Antidifferentiating in Equation (8) and using the condition y(0) = 0 we find

$$y = \frac{m}{k^2}(kv_o + mg)(1-e^{-kt/m}) - \frac{mg}{k}t. \tag{9}$$

We postpone a graph of Equation (9) until further information is developed.

Exercises

6. Starting with Equation (5), carry out in detail the steps that lead to Equations (8) and (9).

7. Calculate $\lim_{t\to\infty} v$. Can you suggest an appropriate name for this quantity?

8. Sketch a graph of v.

2.3 Ascent Time Versus Descent Time

Finding the ascent time t_a of the ball in our new model is rather straightforward. When $t = t_a$ the ball is at its maximum altitude and therefore has a zero velocity. Putting $v = 0$ in Equation (8) and solving, we obtain

$$(10) \qquad t_a = \frac{m}{k} \ln\left(\frac{kv_o + mg}{mg}\right).$$

The descent time t_d is another story. Proceeding directly, we have

$$(11) \qquad t_d = T - t_a$$

where T is the flight duration time. That is, T is that positive value of t such that $y(T) = 0$. Making use of Equation (9) we find

$$(12) \qquad \frac{m}{k^2}(kv_o + mg)(1-e^{-kT/m}) - \frac{mg}{k} T = 0.$$

Equation (12) is an example of what is known as a transcendental equation. No amount of algebraic manipulation of this equation will deliver T to us (even though there are numerical techniques, like Newton's Method, that could be applied to Equation (12) to approximate T to any desired degree of accuracy). We appear to be at a dead end. What to do? Fortunately there is an indirect method of determining the greater of t_a and t_d without actually computing t_d. It goes as follows: we compute $y(2t_a)$, then

$$\text{(i)} \quad t_a > t_d \text{ if } y(2t_a) < 0$$

$$\text{(ii)} \quad t_a = t_d \text{ if } y(2t_a) = 0$$

$$\text{(iii)} \quad t_a < t_d \text{ if } y(2t_a) > 0.$$

Using Equations (9) and (10), letting $c = kv_o + mg$ (so $k > 0 \Rightarrow c > mg$), and performing some algebraic simplification yields

$$(13) \qquad y(2t_a) = \frac{m}{ck^2}[c^2 - m^2g^2 - 2mgc(\ln(c/mg))].$$

Since $\frac{m}{ck^2}$ is positive, the sign of $y(2t_a)$ is determined by the sign of the quantity

$$(14) \qquad Q(c) = c^2 - m^2g^2 - 2mgc(\ln(c/mg)).$$

The sign of Q(c) is not readily apparent and seems, at least at first glance, to depend on c. That this is not the case is shown by the following argument.

Treating c as a variable quantity and differentiating Q(c) twice with respect to c we get

(15) $\quad Q'(c) = 2c - 2mg[\ln(c/mg)+1]$

and

(16) $\quad Q''(c) = 2 - 2mg/c = 2(1-mg/c).$

From Equation (16) we see that $Q''(c) > 0$ for $c > mg$ so $Q'(c)$ is a strictly increasing function on the interval $[mg,\infty)$. But $Q'(mg) = 0$ so $Q'(c) > 0$ for $c > mg$ and therefore Q(c) is also strictly increasing on $[mg,\infty)$. But $Q(mg) = 0$ so $Q(c) > 0$ for $c > mg$. Hence $y(2t_a) > 0$ and we have

A'. The ascent time t_a is less than the descent time t_d.

Exercises

9. Derive Equation (13).

10. Evaluate $\lim_{k\to 0} t_a$. (Compare with Exercise 2.)

11. Sketch the graph of the ascent time versus the drag coefficient.

2.4 Initial Velocity Versus Final Velocity

Looking at the graph of v (Exercise 8) we observe that while the ball is descending its speed is always less than mg/k. Thus, if $v_o \geq mg/k$, the initial speed $|v_o|$ exceeds the final speed $|v_f|$. In this section we shall show that this is always the case, regardless of the value of v_o.

Again we have to abandon the direct aproach ($v_f = v(T)$, but T is unknown) and use an alternative method. Let us return to the first model for a moment. Multiplying Equation (1) by $\dot{y}$ and rearranging terms we get

(17) $\quad m\dot{y}\ddot{y} + mg\dot{y} = 0.$

Notice that the left-hand side of Equation (17) is the derivative of the quantity

(18) $\quad E = \frac{1}{2}m\dot{y}^2 + mgy = \frac{1}{2}mv^2 + mgy.$

E represents the total energy of the ball. It is the sum of the kinetic energy ($(1/2)mv^2$) and potential energy (mgy). Equation (17) tells us that $E'(t) = 0$ which implies that E is constant. This is consistent with the law of conservation of energy which applies to the motion of the ball in this model. This physical law, a consequence of Newton's second law of motion, may be stated as follows: If a body moves in a straight line under the influence of a force which is constant or depends only on the position of the body, then the total energy of the body remains constant during the motion.

As we are assuming the presence of a frictional force in our model, we should expect E to decrease as time goes on. Multiplying Equation (4) by $\dot{y}$ and rearranging terms gives us

$$m\dot{y}\ddot{y} + mg\dot{y} = -k\dot{y}^2 \tag{19}$$

or

$$E'(t) = -k\dot{y}^2. \tag{20}$$

Therefore on the intervals $[0,t_a)$ and $(t_a,T]$ $E'(t) < 0$ so E is indeed a strictly decreasing function on $[0,T]$ which implies

$$E(0) > E(T). \tag{21}$$

But

$$E(0) = \tfrac{1}{2}mv^2(0) + mgy(0) = \tfrac{1}{2}mv_o^2 \tag{22}$$

and

$$E(T) = \tfrac{1}{2}mv^2(T) + mgy(T) = \tfrac{1}{2}mv_f^2. \tag{23}$$

Combining (21), (22), and (23) yields $|v_o| > |v_f|$ or

B'. The initial velocity v_o is greater in magnitude than the final velocity v_f.

If we use Equation (8) in Equation (4) we get

$$m\ddot{y} = (kv_o+mg)e^{-kt/m}. \tag{24}$$

Thus $\ddot{y} < 0$ and the graph of y (Equation (9)) is always concave down. Armed with this bit of information and A' and B' we can now sketch the graph of y as

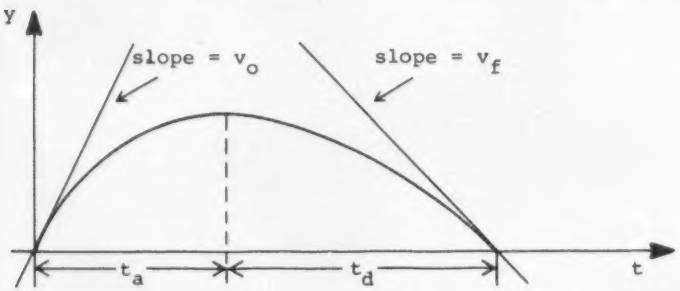


Figure 3. Displacement graph assuming a drag force proportional to the velocity.

3. CONCLUDING REMARKS

3.1 A Comparison of the Two Models

In the second model, unlike the first, pertinent information (such as t_d, T, v_f) is not readily available. However, we are still able to make qualitative statements such as A' and B' and should, of course, have more confidence in their validity as opposed to A and B. From Exercises 2, 10, and 11 we see that the ascent time is greatest in the gravity only model. The flight duration time in the first model is twice the ascent time, or $2v_o/g$. If we put $t = T$, $v = v_f$ in Equation (8) and make use of Equation (12), we find

$$v_f = v_o - gT. \tag{25}$$

Now from B' we have $v_f > -v_o$, so $T < 2v_o/g$. Thus the flight duration time (and consequently the descent time) is also greatest in the gravity only case. In fact, a great deal more can be said.

> The graph in Figure 3 lies wholly below the graph in Figure 1 and the smaller the drag coefficient the closer the graphs match (see Section [S-2] of the Special Assistance Supplement).

Moreover, using Equation (25) and noting that $v_f < 0$ we obtain $T > v_o/g$. Thus we have the (surprising?) result that no matter how overbearing the drag force may be, the flight duration time will always exceed v_o/g.

Exercises

12. Starting with Equations (8) and (12) carry out the steps that lead to Equation (25).

13. Evaluate the following limits:
    a. $\lim_{k\to\infty} t_a$ b. $\lim_{k\to\infty} T$ c. $\lim_{k\to\infty} t_d$ d. $\lim_{k\to\infty} v_f$.

3.2 The Next Step

While model 2 is clearly an improvement over model 1, it should not be regarded as a final statement. There is certainly room for improvement, primarily in the representation of the drag force. Figure 2 seems to indicate that a quadratic approximation to the drag force (i.e., $D = \pm k\dot{y}^2$, $k > 0$, with the minus sign being used for the ascending portion of the flight and the plus sign for the descending portion) might prove to be a better description than the linear second model. Adopting this idea leads to

$$m\ddot{y} = -k\dot{y}^2 - mg \tag{26}$$

and

$$m\ddot{y} = k\dot{y}^2 - mg \tag{27}$$

as the "ascending and descending" equations of motion, respectively. Although the results of the Special Assistance Supplement are of no use in solving these equations, their solutions can be obtained by elementary means. We leave that task and the subsequent analysis to another time.

4. ANSWERS TO EXERCISES

1. Dividing through by m and antidifferentiating we get

$$\dot{y} = -gt + C_1.$$

Putting $t = 0$ and $\dot{y} = v_o$ yields $C_1 = v_o$ and

$$\dot{y} = -gt + v_o.$$

Antidifferentiating once again gives us

$$y = -\tfrac{1}{2}gt^2 + v_o t + C_2.$$

Putting $t = 0$ and $y = 0$ produces $C_2 = 0$ and hence

$$y = -\tfrac{1}{2}gt^2 + v_o t.$$

2. Putting $\dot{y} = 0$ and solving for t yields $t_a = v_o/g$.

3. Putting $t = t_a = v_o/g$ in Equation (2) gives us a maximum height of $v_o^2/2g$.

4. The drag force operates in a direction opposite to the direction of motion.

5. The density of the air and the surface area of the ball.

6. Applying the result of the Special Assistance Supplement to Equation (5) gives us

$$v = Ae^{-kt/m} - \frac{mg}{k}.$$

Putting $t = 0$, $v = v_o$ we get $v_o = A - \frac{mg}{k} \Rightarrow A = v_o + \frac{mg}{k}$

and $v = \frac{1}{k}(kv_o + mg)e^{-kt/m} - \frac{mg}{k}$.

Antidifferentiating yields

$$y = \frac{-m}{k^2}(kv_o + mg)e^{-kt/m} - \frac{mg}{k}t + C.$$

Putting $t = 0$, $y = 0$ we find

$$0 = \frac{-m}{k^2}(kv_o + mg) + C \Rightarrow C = \frac{m}{k^2}(kv_o + mg)$$

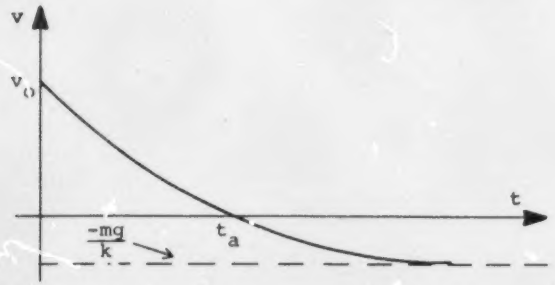
and $y = \frac{-m}{k^2}(kv_o + mg)(1 - e^{-kt/m}) - \frac{mg}{k}t.$

7. We have

$$\lim_{t \to \infty} v = \frac{-mg}{k},$$

which is referred to as the limiting or terminal velocity.

8.

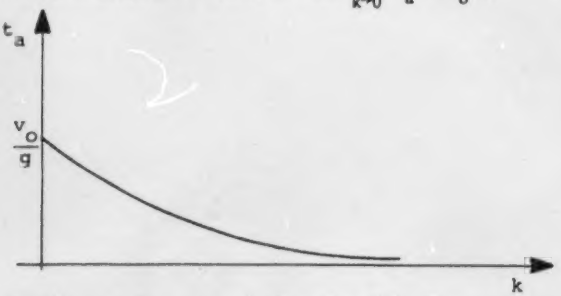


9. In computing $y(2t_a)$ an expression of the form $e^{-2(\ln A)}$ arises, where $A = c/mg$. The thing to realize is that this simplifies to

$$e^{-\ln A^2} = e^{\ln(1/A^2)} = 1/A^2.$$

10. Using L'Hospital's rule we find $\lim_{k\to 0} t_a = v_o/g$.

11.



12. Putting $t = T$, $v = v_f$ in Equation (8) gives us

$$v_f = \frac{1}{k}(kv_o + mg)e^{-kT/m} - \frac{mg}{k}.$$

From Equation (12) we get

$$\frac{1}{k}(kv_o + mg)e^{-kT/m} = \frac{1}{k}(kv_o + mg) - gT = v_o + \frac{mg}{k} - gT.$$

Thus $v_f = v_o - gT$.

13. a. $\lim_{k\to\infty} t_a = 0$ b. $\lim_{k\to\infty} T = v_o/g$ c. $\lim_{k\to\infty} t_d = v_o/g$

d. $\lim_{k\to\infty} v_f = 0$.

SPECIAL ASSISTANCE SUPPLEMENT

[S-1]

Taking the derivative of the function $f(t) = Ae^{kt}$, where A and k are constants, we notice that $f'(t) = kf(t)$ and conclude that $f(t)$ is a solution to the differential equation

$$(I) \qquad \frac{dx}{dt} = kx$$

for all values of the constant A. As to the existence of other solutions of this equation, we offer the following Theorem 1. All solutions of Equation (I) are of the form

$$x = Ae^{kt}.$$

Proof: Suppose $g(t)$ is a solution (i.e., $g'(t) = kg(t)$). Introducing the function $h(t) = g(t)\, e^{-kt}$, we find

$$h'(t) = g'(t)e^{-kt} - kg(t)e^{-kt} = [kg(t)]e^{-kt} - kg(t)e^{-kt} = 0,$$

which implies that $h(t)$ must be a constant function, say having the constant value A. Hence $g(t) = Ae^{kt}$ and the theorem is proven.

Actually, for our purposes we need a slight generalization of Theorem 1, namely

Theorem 2. All solutions of the differential equation

$$(II) \qquad \frac{dx}{dt} = k_1 x + k_2 \qquad (k_1 \neq 0)$$

are of the form $x = Ae^{k_1 t} - k_2/k_1$.

Proof: Suppose $f(t)$ is a solution to Equation (II). Then

$$\frac{d}{dt}\,[f(t)+k_2/k_1] = f'(t) = k_1 f(t) + k_2 = k_1[f(t)+k_2/k_1].$$

Thus the function $[f(t) + k_2/k_1]$ is a solution to Equation (I) ($k = k_1$) so, by Theorem 1, it must be of the form

$$Ae^{k_1 t}.$$

Hence

$$f(t) = Ae^{k_1 t} - k_2/k_1$$

and the theorem is proven.

[S-2]

Reference to quantities associated with model 1 will be subscripted with a 1 and those dealing with model 2 will bear the subscript 2.

To show that the graph in Figure 3 lies below the graph in Figure 1 means showing the difference function

(III) $$d(t) = y_1(t) - y_2(t)$$

to be positive on the interval $(0,T_2)$. Since $d(0) = 0$, one way of accomplishing this is to show that $d(t)$ has a (unique) absolute minimum at $t = 0$. By the Critical Value Theorem the absolute minimum of $d(t)$ on $[0,T_2]$ occurs either at an endpoint or an interior critical point. As $d(T_2) > 0$, we can rule out the right endpoint. Differentiating $d(t)$ twice and making use of Equations (1) and (4) we find

(IV) $$\ddot{d}(t) = (k/m)\dot{y}_2(t).$$

Thus

$$\ddot{d}(t) > 0 \text{ on } (0,t_{a_2})$$

and $\dot{d}(t)$ is strictly increasing on

$$[0,t_{a_2}].$$

But $\dot{d}(0) = 0$ so $\dot{d}(t) > 0$ on $(0,t_{a_2}]$

and $d(t)$ has no critical points in this interval. There is no need to look for critical points in

$$(t_{a_2},T_2)$$

since any such critical points would have to be relative maxima by Equation (IV) and the second derivative test. Therefore $d(t)$ has a unique absolute minimum at $t = 0$ and $d(t) > 0$ on $(0,T_2)$.

To establish the second part of our contention we shall show

$$\lim_{k\to 0} y_2(t) = y_1(t) \text{ for all } t \geq 0.$$

Rewriting Equation (9) as

(V) $$y_2 = \frac{m(kv_0 + mg)(1-e^{-kt/m}) - mgkt}{k^2}.$$

we note that $\lim_{k\to 0} y_2(t)$ is a "0/0" intermediate form and thus a prime candidate for l'Hospital's rule. Differentiating the numerator and denominator of Equation (V) with respect to k we find $\lim_{k\to 0} y_2(t)$ is equal to

$$\text{(VI)} \quad \lim_{k\to 0} \frac{(kv_o+mg)(te^{-kt/m}) + mv_o(1-e^{-kt/m}) - mgt}{2k}.$$

As this is still a "0/0" indeterminate form, we apply l'Hospital's rule once again to get $\lim_{k\to 0} y_2(t)$ equal to

$$\text{(VII)} \quad \lim_{k\to 0} \frac{v_o te^{-kt/m} - (kv_o+mg)(t^2/m)e^{-kt/m} + v_o te^{-kt/m}}{2}$$

or

$$\text{(VIII)} \quad \lim_{k\to 0} y_2(t) = \frac{-gt^2 + 2v_o t}{2} = y_1(t).$$

From the graph of the ascent time versus the drag coefficient (Exercise 11) we see that t_a approaches v_o/g <u>monotonically</u> as k approaches 0. That is, if k_1 and k_2 are two values of the drag coefficient with $k_1 < k_2$, then $t_a(k_1) > t_a(k_2)$. Similar analysis would show that the limit involved in Equation (VIII) is also monotonic or, equivalently, the graph of Equation (9) with $k = k_2$ lies wholly below the graph of Equation (9) with $k = k_1$.